# APPLICATION OF A METHOD OF D'ALEMBERT TO THE PROOF OF STURM'S THEOREMS OF COMPARISON*

BY

MAXIME BÔCHER

Of the many theorems contained in STURM'S famous memoir in the first volume of Liouville's Journal (1836), p. 106, two, which I have called the *Theorems of Comparison*, may be regarded as most fundamental. I have recently shown† how the methods which STURM used for establishing these theorems can be thrown into rigorous form.

In the present paper I propose to prove these theorems by a simpler‡ and more direct method. This method was suggested to me by a passage, to which Professor H. BURKHARDT kindly called my attention, in one of D'ALEMBERT'S papers on the vibration of strings.§ D'ALEMBERT'S fundamental idea, and indeed all that I here preserve of his method, consists in replacing the linear differential equations by RICCATI'S equations.||

## § 1.

We begin then by considering two of RICCATI'S equations:

$$(R_1) \qquad \frac{d\omega}{dx} = A_1(x) + C_1(x)\,\omega^2,$$

$$(R_2) \qquad \frac{d\omega}{dx} = A_2(x) + C_2(x)\,\omega^2,$$

where $A_1$, $A_2$, $C_1$, $C_2$ are, throughout the interval $a \leqq x \leqq b$, continuous functions¶ of $x$ satisfying the inequalities:

$$A_2(x) \geqq A_1(x), \qquad C_2(x) \geqq C_1(x).$$

* Presented to the Society June 29, 1900. Received for publication July 8, 1900.

† Bulletin of the American Mathematical Society, April, 1898, p. 295, and December, 1899, p. 100.

‡ Simpler, at least, if we wish to establish the theorems in all their generality.

§ Memoirs of the Berlin Academy, vol. 70 (1763), p. 242.

|| Sturm also in the paper quoted (p. 159) uses Riccati's equations, but only incidentally, and for quite a different purpose.

¶ All the quantities used in this paper are real.

We assume that $(R_1)$ and $(R_2)$ have solutions, $\omega_1(x)$ and $\omega_2(x)$ respectively, which are continuous throughout the interval $a \leqq x \leqq b$.

LEMMA I. *If* $A_2(c) > A_1(c)$ *and* $\omega_2(c) = \omega_1(c)$ $(a \leqq c < b)$, *then a positive* $\epsilon$ *exists such that*

$$\omega_2(x) > \omega_1(x) \qquad (c < x < c + \epsilon).$$

For by comparing $(R_1)$ and $(R_2)$ we see that:

$$\omega_2'(c) > \omega_1'(c).*$$

Accordingly if $f(x) = \omega_2(x) - \omega_1(x)$, we have $f(c) = 0$, $f'(c) > 0$, from which it follows that, for values of $x$ a little greater than $c$, $f(x) > 0$.

THEOREM I. *If* $\omega_2(a) \geqq \omega_1(a)$, *then* $\omega_2(x) \geqq \omega_1(x)$ $(a < x \leqq b)$.

This theorem we prove first on the supposition that $A_1$ and $A_2$ are not equal at any point of the interval $a \leqq x < b$.

First we notice that the theorem is surely true in a small neighborhood of the point $a$ $(a < x < a + \epsilon)$. This we see at once when $\omega_2(a) > \omega_1(a)$ from the continuity of $\omega_1$ and $\omega_2$, and when $\omega_2(a) = \omega_1(a)$ from lemma I.

Let us now consider the interval $a < x < c$ where $c$ has the largest value $(c \leqq b)$ such that our theorem holds throughout this whole interval. If $c < b$ it is obvious that $\omega_2(c) = \omega_1(c)$, and, therefore, by lemma I $\omega_2(x) > \omega_1(x)$ $(c < x < c + \epsilon)$. This is contrary to hypothesis, as it shows that $c$ might have had a larger value. Therefore $c = b$ and our theorem, in the special case we are considering, is proved.

In order to establish the theorem in the general case we first prove

LEMMA II. *If* $\omega_2(c) = \omega_1(c)$ $(a \leqq c < b)$, *then a positive* $\epsilon$ *exists such that:*

$$\omega_2(x) \geqq \omega_1(x) \qquad (c < x < c + \epsilon).$$

To prove this we consider the new RICCATI'S equation:

$$(\bar{R}_2) \qquad \frac{d\omega}{dx} = A_2(x) + \lambda + C_2(x)\,\omega^2,$$

where $\lambda$, which we shall regard as a parameter, is independent of $x$. Let $\bar{\omega}_2(x, \lambda)$ be the solution of this equation which has at the point $x = c$ the same value which $\omega_2(x)$ has there. If now we restrict $\lambda$ to the interval $0 \leqq \lambda \leqq k$ (where $k$ is any positive constant), there exists a positive $\epsilon$ independent of $\lambda$ such that $\bar{\omega}_2(x, \lambda)$ is a continuous function of $(x, \lambda)$ when $c \leqq x < c + \epsilon$.† In particular we have for every value of $x$ in this interval:

$$\lim_{\lambda=0} \bar{\omega}_2(x, \lambda) = \omega_2(x).$$

---

* We use accents here and in what follows to denote differentiation.

† This follows at once from the analysis on p. 303 of PICARD'S *Traité d'Analyse*, vol. 2.

Let us now compare $\bar{\omega}_2(x, \lambda)\ (\lambda > 0)$ with $\omega_1(x)$ by means of that part of theorem I which we have already proved. This gives

$$\bar{\omega}_2(x, \lambda) > \omega_1(x) \qquad (c < x < c + \varepsilon).$$

By taking the limit for $\lambda = 0$ the truth of our lemma follows at once.

We can now use lemma II to prove theorem I in the general case as we used lemma I to prove it in the special case. As the reasoning is precisely the same, we will not repeat it.

The following modification of theorem I will be of use to us:

THEOREM I′. *If* $\omega_1(b) \geqq \omega_2(b)$ *then* $\omega_1(x) \geqq \omega_2(x)$ $(a \leqq x < b)$.

This theorem may most readily be deduced from theorem I by using the transformation $\bar{x} = a + b - x$ which has the effect of interchanging the points $a$ and $b$ and of changing the sign of $\omega'$.

We will now make theorem I a little more precise as follows:

THEOREM II. *If* $\omega_2(a) \geqq \omega_1(a)$ *then* $\omega_2(x) > \omega_1(x)$ $(a < x \leqq b)$ *provided that when* $\omega_2(a) = \omega_1(a)$ *one excludes the two cases:*

1) *throughout a certain neighborhood of* $a$ $A_2 = A_1$, $C_2 = C_1$.

2) *in every neighborhood of* $a$ *a point* $p$ *exists such that throughout a certain neighborhood of* $p$ $\omega_2 = \omega_1 = 0$.*

We have here merely to prove that if $\omega_2 = \omega_1$ at any point of the interval $a < x < b$, one or the other of the cases here excluded must occur. Let $c$ be a point at which $\omega_2 = \omega_1$. Then theorem I′ tells us that

$$\omega_2(x) \leqq \omega_1(x) \qquad (a < x < c),$$

while by theorem I:

$$\omega_2(x) \geqq \omega_1(x) \qquad (a < x < c).$$

Accordingly:

$$\omega_2(x) = \omega_1(x) \qquad (a < x < c);$$

and since $\omega_1$ and $\omega_2$ are continuous

$$\omega_2(a) = \omega_1(a).$$

Therefore:

$$\omega_2'(x) = \omega_1'(x) \qquad (a \leqq x < c).$$

By subtracting $(R_1)$ from $(R_2)$ we now get:

$$(A) \qquad 0 = A_2(x) - A_1(x) + \big(C_2(x) - C_1(x)\big)\omega_1^2 \qquad (a \leqq x < c).$$

Since $A_2 - A_1$, $C_2 - C_1$, and $\omega_1^2$ can none of them be negative

$$A_2(x) = A_1(x) \qquad (a \leqq x < c).$$

* This second case can obviously occur only when $A_1 = A_2 = 0$ throughout the neighborhood of $p$.

Moreover *either* throughout a certain neighborhood of $a$ $C_2 = C_1$ (in which case we have exception 1) *or* there are points in every neighborhood of $a$ at which $C_2 > C_1$. Throughout a certain neighborhood of any one of these points we must (on account of the continuity of $C_1$ and $C_2$) also have $C_2 > C_1$, and therefore on account of the equality $(A)$ $\omega_1 = \omega_2 = 0$ and we have the exceptional case 2.

## § 2.

Instead of assuming, as we did in the last section, that $\omega_1$ and $\omega_2$ are continuous throughout the interval $a \leqq x \leqq b$, we will now merely assume that they are continuous throughout the interval $a < x \leqq b$, and that they are either continuous at the point $a$ or become positively or negatively infinite there.

THEOREM III. *If* $\omega_2(a) = +\infty$, *then* $\omega_2(x) > \omega_1(x)$ $(a < x \leqq b)$, *provided that; if* $\omega_1(a) = +\infty$, *we exclude the case in which throughout a certain neighborhood of* $a$ $A_2 = A_1$, $C_2 = C_1$.

If we can prove this theorem for the immediate neighborhood of $a$, its truth for the whole interval $a < x \leqq b$ will follow at once by an application of theorem II to the remainder of this interval. Moreover except in the case $\omega_1(a) = +\infty$ the theorem follows for the neighborhood of $a$ from the mere continuity of $\omega_1$ and $\omega_2$.

It remains therefore only to prove the theorem for the neighborhood of $a$ when $\omega_1(a) = +\infty$. For this purpose we take the neighborhood of $a$ so short that neither $\omega_1$ nor $\omega_2$ vanishes in it, and we introduce into the equations $(R_1)$ and $(R_2)$ the new dependent variables:

$$\bar{\omega}_1 = -\frac{1}{\omega_1}, \qquad \bar{\omega}_2 = -\frac{1}{\omega_2},$$

getting as the equations satisfied by $\bar{\omega}_1$ and $\bar{\omega}_2$:

$$\frac{d\bar{\omega}}{dx} = C_1(x) + A_1(x)\cdot\bar{\omega}^2,$$

$$\frac{d\bar{\omega}}{dx} = C_2(x) + A_2(x)\cdot\bar{\omega}^2.$$

These equations having the same form as $(R_1)$ and $(R_2)$, and $\bar{\omega}_1$, $\bar{\omega}_2$ being solutions of them which are continuous in the neighborhood of $a$, we can at once apply theorem II to them. Since $\bar{\omega}_1$ and $\bar{\omega}_2$ do not vanish in the neighborhood of $a$, exception 2 cannot occur. We see thus that, except when throughout a certain neighborhood of $a$ $A_1 = A_2$ and $C_1 = C_2$,

$$\bar{\omega}_2(x) > \bar{\omega}_1(x) \qquad (a < x < a + \varepsilon).$$

Therefore, since $\omega_1$ and $\omega_2$ are both positive:

$$\omega_2(x) > \omega_1(x) \qquad (a < x < a + \varepsilon).$$

Finally, if we drop the requirement that $\omega_1$ and $\omega_2$ be continuous at $b$, theorems II and III will obviously still hold throughout the interval $a < x < b$. We add also the following theorem:

THEOREM IV. *If $\omega_1$ and $\omega_2$ are continuous at $a$ and $\omega_2(a) \geqq \omega_1(a)$, or if $\omega_2(a) = +\infty$ while $\omega_1$ is either continuous at $a$ or becomes positively or negatively infinite there, then, provided we exclude the exceptional cases of theorems II and III, we cannot have $\omega_2(b) = -\infty$.*

For we should also have $\omega_1(b) = -\infty$ on account of the inequality

$$\omega_2(x) > \omega_1(x) \qquad (a<x<b).$$

If then we let as above:

$$\bar{\omega}_1 = -\frac{1}{\omega_1}, \qquad \bar{\omega}_2 = -\frac{1}{\omega_2},$$

in the neighborhood of $b$ these functions are continuous and vanish only at $b$. Since in the neighborhood of $b$ they satisfy the inequality $\bar{\omega}_2 > \bar{\omega}_1$, they should also by theorem II satisfy this inequality at $b$; whereas they are equal there. We are thus led to a contradiction and our theorem is proved.

## § 3.

We now turn to the two linear equations:

$$(L_1) \qquad \frac{d\left(K_1(x)\dfrac{dy}{dx}\right)}{dx} + G_1(x)\,y = 0,$$

$$(L_2) \qquad \frac{d\left(K_2(x)\dfrac{dy}{dx}\right)}{dx} + G_2(x)\,y = 0.$$

Here we assume that $K_1$, $K_2$, $G_1$, $G_2$ are throughout the interval $a \leqq x \leqq b$ continuous functions of $x$ which satisfy the inequalities:

$$K_2(x) \geqq K_1(x) > 0, \qquad G_2(x) \leqq G_1(x).$$

Moreover we assume that $K_1$ and $K_2$ have continuous first derivatives throughout this interval.

Let $y_1(x)$ and $y_2(x)$ be solutions of $(L_1)$ and $(L_2)$ respectively. These functions are, of course, continuous together with their first derivatives throughout the interval $a \leqq x \leqq b$. We shall further assume that if $y_2(a) \neq 0$ then:

$$y_1(a) \neq 0 \quad \text{and} \quad K_2(a)\frac{y_2'(a)}{y_2(a)} \geqq K_1(a)\frac{y_1'(a)}{y_1(a)}.$$

Here again there are two special cases which must be excluded:

1) If $y_2(a)=y_1(a)=0$, or if $y_2(a)\neq 0$ and $K_2(a)y_2'(a)/y_2(a)=K_1(a)y_1'(a)/y_1(a)$, we exclude the case in which throughout a certain neighborhood of $a$ $K_1=K_2$ and $G_1=G_2$.

2) If $y_2'(a)=y_1'(a)=0$, we exclude the case in which in every neighborhood of $a$ a point $p$ exists such that throughout a certain neighborhood of $p$ $y_1'=y_2'=0$. This exception includes the case in which $y_1$ and $y_2$ vanish at all points of the interval $a\leqq x\leqq b$. Apart from this trivial case we note that the case we here exclude can occur only when $G_1$ and $G_2$ both vanish throughout the neighborhoods of the points $p$ in question.*

These restrictions having been made we let:

$$\omega_1=K_1\frac{y_1'}{y_1}, \qquad \omega_2=K_2\frac{y_2'}{y_2}.$$

The functions $\omega_1$ and $\omega_2$ then satisfy the equations $(R_1)$ and $(R_2)$ respectively if

$$A_1=-G_1, \quad A_2=-G_2, \quad C_1=-1/K_1, \quad C_2=-1/K_2.$$

We therefore get at once from theorems II and III:

THEOREM V. *If neither $y_1$ nor $y_2$ vanishes in the interval $a<x<b$ then:*

$$K_2\frac{y_2'}{y_2}>K_1\frac{y_1'}{y_1} \qquad (a<x<b).$$

From theorem IV we get:

THEOREM VI. *If $y_2$ does not vanish in the interval $a<x<b$ and if $y_2(b)=0$, then $y_1$ has at least one root in the interval $a<x<b$.*

## § 4.

The two theorems of the last section are nothing but special cases of Sturm's theorems of comparison, and from them the general theorems will now be deduced.

STURM'S FIRST THEOREM OF COMPARISON. *If $y_2$ has $n$ roots in the interval $a<x\leqq b$, then $y_1$ has at least $n$ roots there, and the $k$th root of $y_1$ measured from $a$ is less than the $k$th root of $y_2$.*

Let $x_1, x_2, \cdots, x_n$ be the roots of $y_2$ $(a<x_1<x_2<\cdots<x_n\leqq b)$. The truth of the theorem follows at once when we notice that by theorem VI in the interval $a<x<x_1$ and also in each of the intervals

$$x_i\leqq x<x_{i+1} \qquad (i=1, 2, \cdots, n-1)$$

there lies at least one root of $y_1$.

* This second exception is not ordinarily mentioned. If $G_1$ and $G_2$ are to be analytic throughout the interval $a<x<b$, this case can arise only if $G_1$ and $G_2$ vanish identically, in which case $y=k$ is a solution of both equations, no matter what values $K_1$ and $K_2$ have.

STURM'S SECOND THEOREM OF COMPARISON. *If $c$ is a point of the interval $a < x < b$ where neither $y_1$ nor $y_2$ is zero, and if in the interval $a < x < c$, $y_1$ and $y_2$ have the same number $n$ of roots, then:*

$$K_2(c)\frac{y_2'(c)}{y_2(c)} > K_1(c)\frac{y_1'(c)}{y_1(c)}.$$

If $n = 0$ this reduces to theorem V. If $n > 0$ let $x_n$ be the $n$th root of $y_2$ measured from $a$. Then since, by STURM'S first theorem of comparison, $y_1$ has at least $n$ roots in the interval $a < x < x_n$, and by hypothesis it has just $n$ roots in the interval $a < x < c$, $y_1$ can have no root in the interval $x_n \leqq x \leqq c$. We may therefore apply theorem V to the interval $x_n < x < c + \epsilon$ and STURM'S theorem follows at once.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.